

Balancing Unicycle Travelling on an Inclined Surface

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| ARTICLE INFO | ABSTRACT |
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| <p>Article History: Received 19 January 2022 Received in revised form 29 February 2022 Accepted 11 June 2022 Available online 14 June 2022</p> | <p>This study focuses on the control of a self-balancing unicycle robot equipped with a roll stabilization mechanism and a single drive wheel responsible for both maintaining upright posture and tracking a predefined path. To achieve stable and responsive behavior, Linear Quadratic Regulators (LQRs) are designed and implemented for roll, pitch, and trajectory tracking control loops. While conventional LQR controllers are generally effective in attenuating external disturbances such as road inclines, they may not fully compensate for the dynamic variations induced by significant slope changes. To address this limitation, a gain-scheduled LQR strategy is proposed, in which controller gains are adaptively adjusted based on varying road inclinations. The road slope alters the dynamic behavior of the unicycle system and introduces external forces that can degrade performance if not adequately compensated. The proposed gain scheduling approach enhances the system's adaptability and robustness, ensuring more accurate path tracking and upright stability under non-uniform terrain conditions. A comparative analysis is conducted between the gain-scheduled LQR and a standard fixed-gain LQR design. Simulation results demonstrate the effectiveness of the proposed method in improving the robot's performance across varying slope conditions, validating its potential for real-world application in unstructured or sloped environments.</p> |
| <p>Keywords: Self-Balancing Robot, Unicycle Robot, LQR, Slope Climbing</p> | |

1. INTRODUCTION

Unicycle robot is preferred to multi-wheeled ones due to its higher degree of mobility and less needed space. However, it is inherently unstable and its roll and pitch stabilization bring up many challenging problems [1]. Several unicycle configurations have been developed among them is one that consists of a body, a reaction disk for roll balancing and an in-wheel motor for both forward-backward moving and pitch stabilization. Balancing a unicycle means keeping it in upright position despite of surrounding disturbances and its required motion to perform its intended tasks.

In early works, the pitch and roll are considered decoupled and basic PD controller has been used. Then, plenty of algorithms such as Linear Quadratic Regulator (LQR), nonlinear and intelligent methods have been adopted to stabilize unicycle in its vertical posture [2]. Based on the decoupled scenario, a LQR for pitch and fuzzy sliding

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mode for roll has been implemented in [3] for a unicycle moving on a level surface. The fuzzy sliding is aimed to manage the ignored coupling that there exists in the dynamic. Application of gain-scheduling for attaining robustness in the control of unicycle is the focus of exploration in [4]. Researchers in [5] build on integral sliding mode to improve the unicycle balance performance during mobile mission. Similar research has also been reported in [6] to handle disturbance and uncertainty in the model. Gain-scheduling Optimal Fuzzy Logic Controller is investigated in [7] for unicycle control. Keeping unicycle balanced while moving on an inclined surface is examined in [8]. The adopted controllers are LQR and the robot can maintain its posture stabilized on the slopes angles varying from -11° to 11° .

In this study, balancing of a unicycle moving on an inclined surface with -30° to 30° angle is investigated. The vehicle is required to follow a predefined target point on time. Roll is stabilized by a LQR controller and gain scheduling LQR (GS-LQR) is studied for the pitch administration. The simulation results indicate that there is not strong coupling between roll and pitch at the speed set for a slow moving unicycle. Road inclination alters the vehicle dynamic and affects its performance, but LQR by itself with appropriate choice of design weighting can effectively reduce the road tilt impact. However, the road disturbance is better rejected when GS LQR is incorporated.

In section 2, the system description is provided. The control techniques are briefed in section 3. Simulation studies are detailed in section 4 and lastly the conclusion comes in section 5.

2. SYSTEM DESCRIPTION

A version of unicycle robot, studied here, consists of three parts: The lower rotating wheel that manages the unicycle movement and its pitch stabilization, the middle body frame, and the upper disk stabilizing roll as shown in Fig. 1.

The dynamic model of the unicycle mobile robot in generic nonlinear format is expressed by,

$$\begin{aligned} \dot{\underline{x}} &= f(x) + g(x)\underline{u} \\ \underline{x} &= [\theta_d \quad \theta_r \quad \omega_d \quad \omega_r \quad \theta_w \quad \omega_w \quad \omega_p]^T \\ \underline{u} &= [\Gamma_d \quad \Gamma_w]^T \end{aligned} \tag{1}$$

where θ stands for angle and ω for the rotational speed. The subscripts w , d , r and p denote wheel, disk, roll and pitch related rotations, respectively. The wheel and the disk are derived by motors excited by DC voltages generating Γ torques.

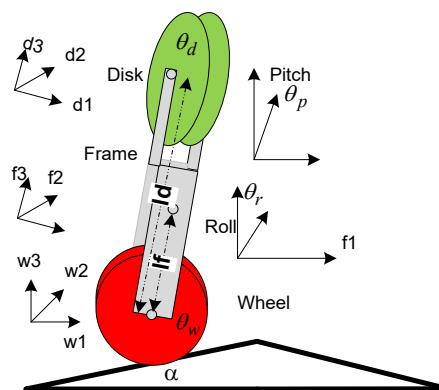


Fig. 1. Model of the unicycle robot [8]

The detailed dynamical model is derived using Lagrangian method as detailed below.

2.1. Roll Model

The unicycle roll dynamics is obtained observing it as a reaction-wheel pendulum [9, 10] and using the coordinate systems depicted in Fig. 1 as follows [8],

$$\begin{aligned} \dot{\theta}_r &= \omega_r, & \dot{\theta}_d &= \omega_d \\ a_2 \dot{\omega}_r + I_{d1} \cos \theta_p \dot{\omega}_d &= c_2 \\ I_{d1} \dot{\omega}_r + I_{d1} \cos \theta_p \dot{\omega}_d &= I_{d1} \omega_r \omega_p \sin \theta_p + \Gamma_d \end{aligned} \quad (2)$$

with the following parameters,

$$\begin{aligned} a_2 &= [I_{w1} + m_c \cos^2 \alpha + 2m_A r_w \cos \theta_p + (I_{f1} + I_{d1} + m_B) \cos^2 \theta_p + (I_{f1} + I_{d1}) \sin^2 \theta_p] \\ c_2 &= m_T g r_w \sin \theta_r \cos \alpha + m_A g \cos \theta_p \sin \theta_r + \omega_p \sin \theta_p * \{2m_A r_w + 2(I_A + m_B) \cos(\theta_p)\} \omega_r + I_{d1} \omega_d \} \\ m_T &= m_w + m_f + m_d, m_A = m_f l_f + m_d l_d, m_B = m_f l_f^2 + m_d l_d^2, I_A = I_{f1} - I_{f3} + I_{d1} - I_{d3} \end{aligned} \quad (3)$$

where m_d , m_f and m_w are the mass of the disk, the robot body, and the rotating wheel. l_f and l_d are the wheel-frame centroids and the wheel-disk centroids distances. R_p and r_w are the disk and the wheel radiuses. I_{x1} ($x \in [d, f, w]$) represents the moment of inertia along the x_1 axes. α is the road angle and Γ_d is the disk torque. c_2 reflect coupling from the pitch on the roll.

2.2. Pitch Model

The pitch dynamic model is derived considering the robot as an inverted pendulum as follows [8, 1],

$$\begin{aligned} \dot{\theta}_w &= \omega_w, & \dot{\theta}_p &= \omega_p \\ b_1 \dot{\omega}_p + a_1 \dot{\omega}_w &= m_A r_w \sin \theta_p \cos \alpha \omega_p^2 + \Gamma_w - \Gamma_r \\ (I_{f2} + I_{d2} + m_B) \dot{\omega}_p + b_1 \dot{\omega}_w &= c_1 - \Gamma_w + \Gamma_r \end{aligned} \quad (4)$$

with the following parameters,

$$\begin{aligned} c_1 &= \sin \theta_p [m_A g \cos \theta_r - [m_A r_w \cos \alpha + (I_A + m_B) \cos \theta_p] \omega_r^2 - I_{d1} \omega_r \omega_d] \\ a_1 &= (I_{w2} + m_T r_w^2 \cos^2 \alpha + m_T r_w \sin |\alpha|) \\ b_1 &= m_A r_w \cos \theta_p \cos \alpha \end{aligned} \quad (5)$$

The coupling from the roll emerges in the c_1 parameter. Γ_w is the wheel generated torque and Γ_r stands for the torque induced by the road slope.

3. DESIGN OF THE CONTROLLERS

3.1. LQR Controller

Considering (1), a LQR controller is designed based on its linearized from. The linearization of (1) around an operating point $[x_0, u_0]$ renders,

$$\dot{x} = Ax + Bu \quad (6)$$

where $x = x - x_0$ and $u = u - u_0$ are the deviations of the variables from their desired working points. The system will enjoy the advantages of feedback when $u = -Kx$ is applied [11]. This is a full state feedback control strategy. The

closed loop system stability and transient behavior is set through assigning appropriate poles to the system, i.e. the eigenvalues of the matrix $A-BK$ which are tuned by the K values.

Picking strategy for computing K depends on the control objectives, system uncertainty and disturbances. In basic optimal control, K is calculated in a way that the next quadratic objective function is minimized,

$$J = \int_{t_0}^{t_f} [x^T Q x + u^T R u] dt \quad (7)$$

where, t_f is the final time, and t_0 is the initial time. The diagonal positive definite matrixes Q and R are valued using the Bryson's rule [11],

$$Q_{ii} = \frac{1}{\max x_i^2}, \quad R_{jj} = \frac{1}{\max u_j^2}, \quad j = 1, 2, \dots, n \quad (8)$$

and their appropriate values are tuned by trial and error for attaining an optimized transient performance: i.e. the fastest transient at less power consumption.

Solution to the optimal control problem (4) is given by the next algebraic Ricatti equation [12],

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \quad (9)$$

where the symmetric positive semi-definite matrix P is calculated. Then, the optimal control signal is expressed by,

$$u = -Kx = -R^{-1}B^T Px \quad (10)$$

3.2. GS LQR Controller

In real, the unicycle movement is affected by the slope of its path. Considering the road slope, the linearized system is expressed by,

$$\dot{x} = A(\alpha)x + Bu \quad (11)$$

By regarding α as a scheduling variable, a controller is derived that takes into account the road gradient as follows,

$$P(\alpha)A(\alpha) + A(\alpha)^T P(\alpha) - P(\alpha)BR^{-1}B^T P(\alpha) + Q = 0 \quad (12)$$

By solving (7), the following road angle dependent control action is produced,

$$u = -Kx = -R^{-1}B^T P(\alpha)x \quad (13)$$

4. SIMULATION AND ANALYSIS OF THE RESULTS

In this paper, the performance of the unicycle with the specifications depicted in Table 1 is discussed [3]. The simulation consists of a series of motions starting with a 20 seconds motion at speed of 0.5 m/s followed by a 10 seconds stop at the inclined road and then coming down with 0.5m/s speed to the level surface. Various road slopes are examined.

4.1. Roll LQR Control

Design of a roll LQR controller is carried out by linearizing the nonlinear dynamic system (2) around the operating point, $x=[0\ 0\ 0\ 0]$ which leads to the following linear system,

$$\dot{x} = Ax + B\Gamma_d, \quad y = Cx, \quad x = [\theta_r \quad \theta_d \quad \omega_r \quad \omega_d]^T$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 15.56 & 0 & 0 & 0 \\ -15.56 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.75 \\ -0.25.63 \end{bmatrix}$$

Choice of Q and R (4) has significant impact on the transient behavior of the system which is usually tuned by trial and error. The following choice is recognized to be appropriate for this design,

$$Q_{roll} = \text{diag}([100, 0.001, 1, 0.01]), \quad R=1$$

By using (5) and (6), the following control action for the roll is obtained, $K_{roll} = [74.1955\ 0.0316\ 18.7847\ 0.1287]$.

The result of the roll control has been depicted in Fig. (2), where a disturbance of 40° in size is applied at the 15th seconds of the simulation. The designed controller damps the disturbance and maintains the required stability as it is required.

Table 1. Unicycle Robot Parameters

| Symbol | Parameter | | Value |
|----------|-----------------------------------------------------|---------------------|--------|
| m_d | Mass (Kg) | Disk | 1.225 |
| m_f | | Frame | 3.664 |
| m_w | | Wheel | 1.300 |
| r_w | Radius(m) | Wheel | 0.110 |
| R_p | | Disk | 0.200 |
| l_f | The centroids distances(m) | wheel-frame | 0.285 |
| l_d | | wheel-disk | 0.330 |
| I_{w1} | Moment of inertia of the wheel (kg m ²) | W ₁ axis | 0.0038 |
| I_{w2} | | W ₂ axis | 0.0075 |
| I_{w3} | | W ₃ axis | 0.0038 |
| I_{F1} | the frame (kg m ²) | F ₁ axis | 0.045 |
| I_{F2} | | F ₂ axis | 0.045 |
| I_{F3} | | F ₃ axis | 0.005 |
| I_{d1} | the disk(kg m ²) | D ₁ axis | 0.0042 |
| I_{d2} | | D ₂ axis | 0.0038 |
| I_{d3} | | D ₃ axis | 0.0038 |

4.2. Pitch LQR Control

Similarly a pitch LQR controller is also designed by linearizing (3) around the operating point $x=[0\ 0\ 0\ 0]$ which results in the following state space linear system,

$$\dot{x} = Ax + B\Gamma_w, \quad y = Cx, \quad x = [\theta_p \quad \theta_w \quad \omega_p \quad \omega_w]^T$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 18.09 & 0 & 0 \\ 0 & -38.72 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -3.59 \\ 19.75 \end{bmatrix}$$

By assigning,

$$Q_{pitch} = \text{diag}([100, 10, 1, 0.1]), \quad R=1$$

and using (5) and (6), the following optimal controller gain is obtained,

$$K_{pitch} = [-71.0800 \quad -3.1623 \quad -17.2436 \quad -2.7631].$$

The results of the unicycle pitch control performing the aforementioned moving plan are evaluated. The pitch motion, the wheel speed and the wheel speed reference command has been exhibited in Fig.(6). The unicycle wheel speed follows the command with some overshoot and undershoots. Besides, damping of the pitch variations due to the forward motion, cessation, and backward motion have also been demonstrated.

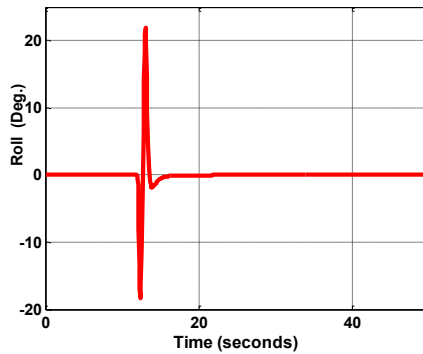


Fig. 2. Roll angle LQR disturbance rejection performance

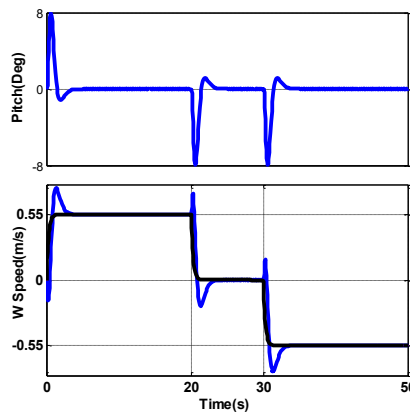


Fig. 3. Pitch and wheel speed following performances

The pitch and path tracking control configuration has been shown in Fig. 4. As it is shown later, the system dynamic has effective zeros that disturb the design merits calculated based on the pole positioning method. Hence, a prefilter is incorporated in the design to issue smoother control command not to strongly excite zeros and to prevent high level abrupt change in the control signal as it has been portrayed in Fig.(5)

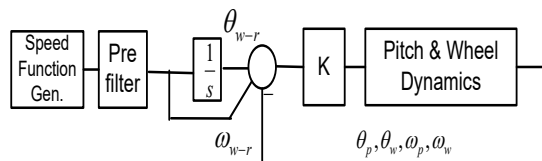


Fig. 4. The complete pitch control configuration

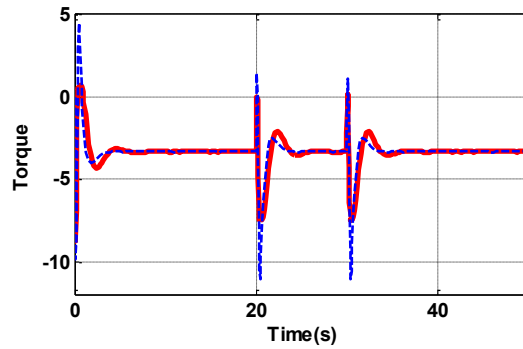


Fig. 5. The control effort size reduction using a prefilter

Meanwhile, the unicycle follows the position command as it is shown in Fig. (6), but steady state error is exhibited. The error turns to zero if the speed reference command is also applied to the control loop as shown in Fig. 4. Now, an integral action is added to the control loop forcing steady state error to diminish, of course at the cost of higher control effort.

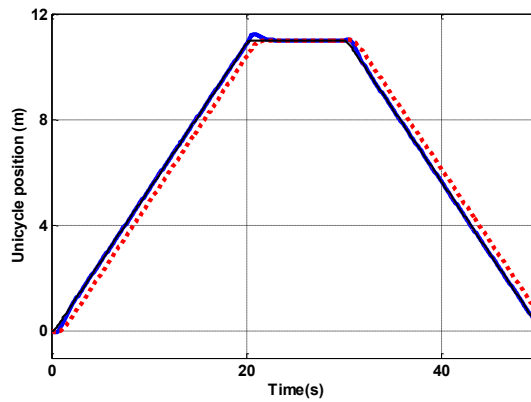


Fig. 6. The unicycle reference path and the trajectory without (dotted line) and with (solid line) employing the wheel speed reference

The performance of the pitch control in disturbance rejection has also been shown in Fig. (6) which the system successfully contains an abrupt 20° in size pulse impact on the pitch.

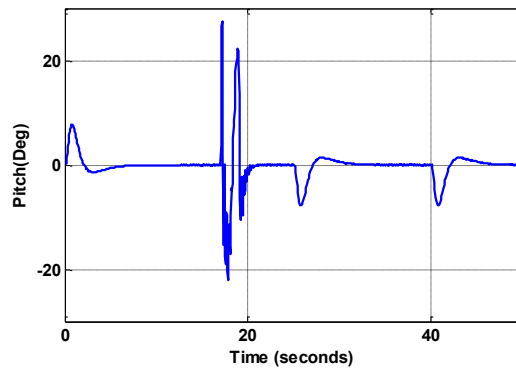


Fig. 7. Pitch LQR disturbance rejection

4.3. Pitch gain scheduling LQR

In this section, the performance of the controller in facing an inclined surface disturbance is analyzed. The system wheel torque, pitch angle and wheel speed have been shown in Fig. (7) where the vehicle travels on a road with 30°, 15° and 0° angles. The vehicle goes up the slope, stops for a while and then comes down. There is a DC level in the control signal which compensates the inertial force trying to drive the unicycle down the inclined road.

To study the effect of the road angle on the unicycle pitch dynamics (3), it is linearized around the null operating point as a function of the road slope, α ,

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_1 m_A g}{b_1^2 - b_2 b_3} & 0 & 0 \\ 0 & \frac{-b_2 m_A g}{b_1^2 - b_2 b_3} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_w \\ \theta_p \\ \omega_w \\ \omega_p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{-(b_1 + b_3)}{b_1^2 - b_2 b_3} \\ \frac{b_1 + b_2}{b_1^2 - b_2 b_3} \end{bmatrix} \Gamma_w \quad (14)$$

$$b_2 = I_{w2} + m_T r_w^2 \cos^2 \alpha + m_T r_w \sin \alpha, \quad b_1 = m_A r_w \cos \alpha, \quad b_3 = I_{f2} + I_{d2} + m_B$$

The wheel location and the pitch angle transfer function is obtained as below,

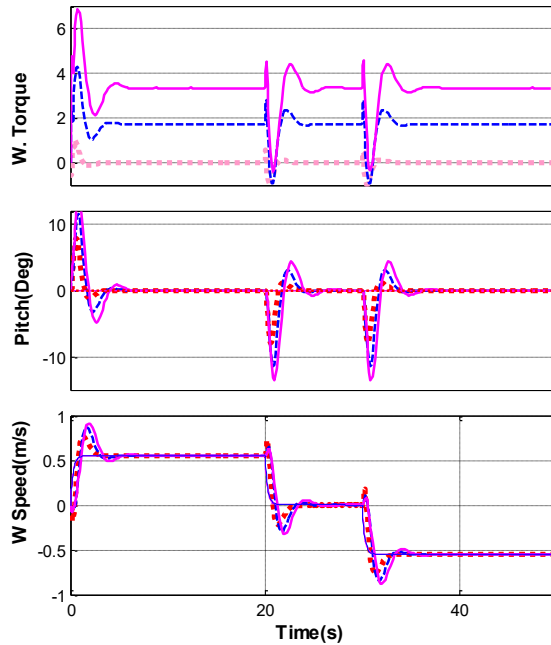


Fig. 8. The wheel speed, pitch and wheel torque curves for the unicycle travelling on a zero (dotted line), 15° (dashed) and 30° (solid line)

$$\frac{\theta_w}{\Gamma_w} = \frac{k_w (s+4)(s-4)}{s^2 (s+p)(s-p)} \quad (15)$$

$$\frac{\theta_p}{\Gamma_w} = \frac{k_p s^2}{s^2 (s+p)(s-p)}$$

Both have two fixed zeros, two fixed poles and two road slope dependent poles, p . Besides, the gain variation is also observed as given below,

$$\alpha = 0 \dots 30^\circ \Rightarrow \begin{cases} p = 6 \dots 4.2 \\ k_w = 28.79 \dots 3.15 \\ k_p = -7 \dots -1.6 \end{cases}$$

The system zeros, especially one in the right half plane which makes the system non minimum phase drastically degrades the intended design performance since the design does not take them into account. For cancelling the effective zeros, different remedies are generally advised such as using prefilter that is included in the control structure.

By appropriate choice of $Q(4)$, the perturbing effect of the road angle can be partially subdued, i.e. where $Q_2 = \text{diag}([100 \ 10 \ 1 \ 0.1])$ is adopted with 0.1 weight for the wheel speed reference command following, the system is not vulnerable to the road slope as exact wheel speed tracking is almost ignored. In such a circumstance the LQR gains are almost the same for all slopes as it has been depicted in Table 2 for Q_1 . Three of the four parameters are not affected and only there is a %10 percent variation in the first one. The result of the simulation using fixed versus GS LQR has been shown in Fig. 8, where almost similar %10 percent improvement in the transient behavior is observed. Alternatively if $Q_2 = \text{diag}([100 \ 10 \ 1 \ 10])$

is picked which gives more weight to the wheel speed tracking, variation in the LQR gain as a function of road slope is increased as it has been depicted in Table 2 for Q_2 .

It is concluded if slow transient response is designed a fix gain LQR can effectively manage the road slope, however if a fast transient response is desired, GS-LQR benefits the system travelling quality.

Table 2. The Lqr Gains for Various Slopes

| | Slope | θ_p | θ_w | ω_p | ω_w |
|-----------|-------|------------|------------|------------|------------|
| Q1 | 0 | -71 | -3.16 | -17 | -2.7 |
| | 15 | -68 | -3.16 | -16.7 | -2.9 |
| | 30 | -65 | -3.16 | -15.56 | -3.4 |
| Q2 | 0 | -136 | -3.16 | -34 | -4.8 |
| | 15 | -120 | -3.16 | -30 | -4.8 |
| | 30 | -90 | -3.16 | -21 | -5.1 |

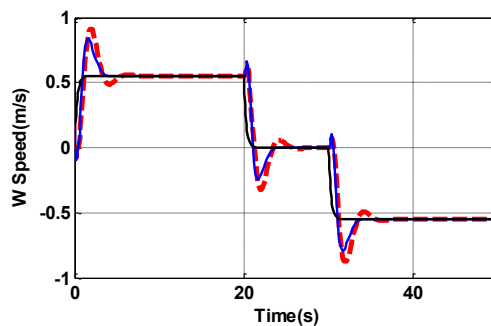


Fig. 9. Improvement in performance using Gain Scheduling LQR controller

5. CONCLUSION

In this paper, the balancing of a unicycle moving on an inclined surface is studied. An LQR controls the roll motion and a GS- LQR manages the pitch and path following. The road angle affects the system dynamic. Depending on the choice of the Q and R weighting parameters of the optimal control, the pitch and the wheel motion will be more or less vulnerable to the road angle. For pitch GS LQR is adopted which improves the system performance.

6. ACKNOWLEDGEMENT

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