Transactions on Machine Intelligence 7(4) (2024) 238-245

ISSN Online: 2821-1693

Transactions on Machine Intelligence

Journal Homepage: https://tmachineintelligence.ir/

A Review of Different Approaches on Determination of Convergence Control Parameter in Homotopy Analysis Method

M. Jalili [1,*]

[1] Department of Mathematics, Neyshabur Branch, Islamic Azad University, Neyshabur, Iran.

ARTICLE INFO

Article History:
Received 30 June 2024
Received in revised form 5 September 2024
Accepted 19 November 2024
Available online 3 December 2024

Keywords:
Homotopy Analysis Method;
Optimal Control Parameter;
Nonlinear Problems

ABSTRACT

One of the principal components of the Homotopy Analysis Method (HAM) is the determination of the convergence control parameter, which plays a pivotal role in ensuring the accuracy and efficiency of solutions obtained using HAM. The convergence control parameter directly impacts the rate of convergence and the precision of the method, making its proper determination essential for solving nonlinear problems. This study aims to systematically compare the performance of several approaches for determining the convergence control parameter in HAM. By examining different methods, the paper highlights their respective strengths, weaknesses, and applicability to a range of nonlinear problems. Numerical experiments and theoretical analysis are conducted to assess the accuracy and convergence rate associated with each approach. Particular attention is given to identifying strategies that achieve a balance between computational efficiency and solution precision. The results provide valuable insights into the impact of the convergence control parameter on HAM's performance and offer guidelines for selecting the most suitable approach for various types of problems. This study contributes to advancing the application of HAM in solving nonlinear equations, enhancing its utility in scientific and engineering contexts.

1. INTRODUCTION

In real-life scenarios, a multitude of scientific and technical challenges across disciplines such as physics, engineering, chemistry, biology, and other fields are governed by nonlinear equations. Unlike linear equations, solving nonlinear equations presents significant complexities due to their intricate nature. The rising interest in mathematical models addressing nonlinear phenomena has driven the development and refinement of both analytical and numerical methods to tackle these equations. Among these methods, the Homotopy Analysis Method (HAM) stands out as a versatile and powerful computational tool that provides highly accurate approximate solutions for a broad spectrum of nonlinear problems.

HAM has been extensively employed to address nonlinear equations under diverse physical and mathematical conditions, encompassing ordinary and partial differential equations, integral and integro-differential equations,

* Corresponding author: Jalili.maryam@yahoo.com
Department of Mathematics, Neyshabur Branch, Islamic Azad University, Neyshabur, Iran

http://dx.doi.org/10.47176/TMI.2024.238

systems of equations, fractional forms, and other related families. Its distinctive feature lies in its ability to construct approximate analytical solutions while maintaining flexibility and adaptability through the use of auxiliary parameters and functions. This flexibility allows researchers to control both the convergence region and the rate of convergence, leading to rapid and reliable results.

Over the years, researchers have contributed to HAM's evolution by providing convergence analyses, refining the method, improving its performance, and simplifying its implementation. The method's efficiency, reliability, and ability to yield rapidly convergent series solutions make it a time-saving alternative for addressing complex nonlinear problems. In this study, we undertake a comprehensive review and analysis of various approaches to HAM, comparing their accuracy, computational efficiency, simplicity, and applicability to diverse nonlinear problems. Through this comparative analysis, we aim to identify the strengths and limitations of different techniques, ultimately offering insights for the broader scientific community.

2. RESEARCH BACKGROUND

The Homotopy Analysis Method (HAM) is recognized as a robust analytical approach for solving nonlinear differential equations, including fractional integro-differential equations (FIDEs) and boundary value problems. Central to the efficacy of HAM is the convergence control parameter, which significantly influences the stability and accuracy of the solutions. Several researchers have contributed to the development and optimization of HAM by investigating methods for determining the convergence control parameter.

Abbasbandy et al. (2013) highlighted the importance of selecting an appropriate convergence control parameter when applying HAM to fractional integro-differential equations. Their research demonstrated that improper parameter selection could adversely affect the convergence of solutions, emphasizing the need for optimized approaches [1]. Similarly, Turkyilmazoglu (2016) introduced a novel strategy for determining the optimal convergence control parameter, focusing on enhancing accuracy and reducing computational effort compared to traditional methods such as constant h-curves [2].

Advances in convergence methods have also been explored through the q-Homotopy Analysis Method (q-HAM), as demonstrated by El-Tawil and Huseen (2013). Their work underscored the conditions under which convergence is guaranteed, enabling more precise adjustments to the control parameter [3]. Hassan and Rashidi (2014) expanded on this by combining HAM with a modified Laplace transform algorithm, allowing for more efficient control of the convergence region [4].

The practical applications of HAM further validate its significance. For instance, Yin et al. (2015) applied a modified version of HAM to solve fractional-order SEIR epidemic models, illustrating how the convergence control parameter ensures reliable outcomes in real-world problems [5]. Akinyemi et al. (2021) discussed the role of convergence in solving von Kármán equations, emphasizing the importance of accurately determining the control parameter for rapid and precise results [6].

Despite the substantial advancements in determining the convergence control parameter, knowledge gaps remain. Research continues to focus on specific applications or equation types, with limited generalization across diverse nonlinear problems. Future efforts may involve integrating machine learning techniques to dynamically optimize the convergence control parameter, ensuring applicability to a broader range of equations.

3. A QUICK GLANCE OF HAM

The description and main ideas of the HAM, that allows us to obtain approximate series solutions to a wide variety of nonlinear differential equations. In this section, we present a brief review of the HAM. by considering the case of a nonlinear equation written in the general

$$N[u(x)] = 0, \tag{1}$$

where N is a nonlinear operator, $u(x)$ is an unknown function. In order to obtain a convergent series solution for the given non-linear problem (1), the HAM is constructed, using $q \in [0,1]$ as an embedding parameter, the so-called zeroth-order deformation equation becomes:

$$(1-q)L[\varphi(x,q)-u_0(x)] = qc_0H(x)N[\varphi(x,q)], \quad (2)$$

where c_0 is the convergence-control parameter, $u_0(x)$ is an initial guess, $H(x)$ is an auxiliary function and L is an auxiliary linear operator which satisfies $L[f]=0$. From (2), when $q=0$ and $q=1$, we have $\varphi(x,0)=u_0(x)$ and $\varphi(x,1)=u(x)$, respectively. Therefore, $\varphi(x,q)$ varies from the initial value $u_0(x)$ to the exact solution $u(x)$ as the embedding parameter is varied from 0 to 1. The fundamental assumption of this algorithm is that the solution of Eq. (2.2) can be expressed as a power series in q of the form,

$$\varphi(x,q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x)q^m, \quad (3)$$

where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x,q)}{\partial q^m}\Big|_{q=0}.$$

And $u_m(x)$ $(m>1)$ are obtained by means of solving the high-order deformation equations,

$$L[u_m(x)-\chi_m u_{m-1}(x)] = c_0H(x)\Re_m(\vec{u}_{m-1}(x)).$$

Defining the vector

$$\vec{u}_n(x) = (u_0(x),\cdots,u_n(x)),$$

and

$$\Re_m(\vec{u}_{m-1}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}[\varphi(x,q)]}{\partial q^{m-1}}\Big|_{q=0},$$

where

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & otherwise \end{cases}$$

the so-called homotopy series solution

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x), \quad (4)$$

which must be one of the solutions of original nonlinear equation (1).

4. DIFFERENT APPROACHES FOR DETERMINING OF CONVERGENCE CONTROL PARAMETER IN HAM

In the classical HAM, drawing c_0-curves determines the range of the convergence-control parameter. A variety of methods were investigated to calculate the optimal convergence-control parameter to get the faster convergence of analytical series solutions obtained for the problem. Assume that the nonlinear equation $N[u(x)] = 0$ is defined on the domain Ω, and $\sum_{k=0}^{m} u_k(x)$ is the m-th order of HAM approximation solution of the governing equation.

In the following, some practical approaches are presented.

4.1. Exact square residual error

In 2007, an optimal method so-called "Optimization Method," is suggested by Yabushita et al. [7] for estimating the optimal convergence control parameters. The exact square residual errors $\Delta_m(c_0)$, is defined as

$$\Delta_m(c_0) = \int_\Omega N[\sum_{k=0}^{m} u_k(x)]^2 dx, \tag{5}$$

that includes an unknown auxiliary parameter c_0, $u_k(x)$ is the k-th order Homotopy approximation. The optimal parameter value of $c_0$is given by minimizing the Eq. (5), which corresponds to the non-linear algebraic equations.

4.2. Average square residual error

The above method is required a significant amount of CPU time to calculate residual error even if the approximation order is low. In order to reduce CPU time, in 2010, Liao [8] proposed an approach so-called "average square residual error," method, which is a discrete approach to computing residuals. For the m-th order HAM approximation is defined as follows

$$E_m(c_0) = \frac{1}{K+1}\sum_{i=0}^{K} N[\sum_{k=0}^{m} u_k(x_i)]^2, \tag{6}$$

where $x_i = i\Delta x \in \Omega, i = 0,1,2,3,\dots\dots\dots, K$, denote the properly chosen $(K+1)$ points. on the domain Ω, and $\Delta x = \frac{length\ of\ \Omega}{K}$.

4.3. L_2-norm method

In 2013, Abbasbandy and Jalili [9] constructed the residual calculating method so-called " L_2-Norm method ", which is also a discrete method and takes less CPU time. Let

$$\Delta = \{a = x_1 < x_2 < x_3 < \dots\dots < x_{n+1} = b\}$$

be a partition of the interval $[a, b]$ and also let

$$\psi_m(x, c_0) = N[x, u_m(x, c_0)] .$$

The L_2 - Norm is defined as

$$\Delta_m(c_0) = \left\|\psi_m(x_1, c_0), \dots\dots\dots, \psi_m(x_{n+1}, c_0)\right\|_2 \tag{7}$$

Thereafter, an optimal value of convergence-control parameter c_0 can be computed by the minimizing of $\Delta_m(c_0)$, corresponding to the non-linear algebraic equation $\frac{d\Delta_m(c_0)}{dc_0} = 0$.

4.4. Optimum value from the ratio

In 2016, Turkyilmazoglu [10], was suggested by letting the ratio

$$\beta = \frac{\| u_{k+1}(x) \|}{\| u_k(x) \|}, \tag{8}$$

with the norm being understood as $L^p - norm(p = 1\ or\ p = 2)$, to be sufficiently small. In other words, for a prescribed c_0, if the ratio is less than unity, then the convergence of HAM is guaranteed. An optimal value for the convergence control parameter c_0 might be as a consequence of $\frac{d\beta}{dC_0} = 0$.

Considering integrals in the ratio

$$\beta = \frac{\int_\Omega u^p{}_{k+1}(x)dt}{\int_\Omega u^p{}_k(x)dt},$$

or its discrete counterpart

$$\beta \approx \frac{\sum_{j=0}^{N}[u_{k+1}(x_j)]^p}{\sum_{j=0}^{N}[u_k(x_j)]^p},$$

brings a more convenient way of evaluating the optimal convergence control parameter c_0.

4.5. Mini batch gradient descent (MBGD)

Tonghui and Yinping[11] in 2022, applied the MBGD in the machine learning method. The unknown parameters in the HAM are optimized by using the method. Let $\theta = \{c_0, \omega\}, C_0$ is the convergence control parameter, ω represents the linear auxiliary operator L , the initial guess solution $u_0(x)$, the auxiliary function $H(x)$ are used to accelerate convergence and control the convergence range. This set may also contain other unknown parameters. $N[u(x_0,\theta)]^2$ is used as the square of the measure margin. The analytical approximate solution $u(x_0,\theta)$ tends to $u(x_0)$, when $N[u(x_0,\theta)]^2 \to 0$, there is $u(x_0,\theta) \to u(x_0)$. Therefore, by minimizing $\int N[u(x,\theta)]^2 dx$, the θ is obtained, which can make the analytical approximate solution converge faster and better.

Definition objective function

$$J(x,\theta) = N[u(x,\theta)]^2 \tag{9}$$

where $x \in \Omega$, and Ω is the value set of the training set, which is a proper subset of the domain of definition.

Then the problem is transformed into the optimization of θ in formula (9), considering that using the ReLU (Rectified Linear Unit) function for gradient truncation to solve the gradient explosion problem will lead to uneven distribution of training data, and the function requires weak adaptability of parameters, Tonghui and Yinping [5], used the Adam gradient descent method to avoid this problem, and it also speeds up the convergence while solving the gradient explosion problem, as shown in Algorithm 1.

Algorithm 1. Gradient descent method

Input: training set X
Output: parameters θ after iterative optimization
1: Initialize learning parameters $\rho_1 = 0.9, \rho_2 = 0.999, \delta = 10^{-8}$,
iteration parameter $k = 1, s_0 = 0, r_0 = 0$, maximum number of iterations K
2: Generate a random point dataset $X \in \Omega$
3: Set the learning rate α, the initial parameter θ_0
4: **Repeat**
5: Randomly select a subset of the training set $X_k = \{x_1, x_2, \cdots, x_n\}$, $X_k \subseteq X$
6: Calculate the gradient: $g = \frac{1}{n}\sum_{i=1}^{n} \nabla_{\theta_k} J(x_i, \theta_k)$
7: Calculate the first and second moments $s_{k+1} = \rho_1 s_k + (1-\rho_1)g, r_{k+1} = \rho_2 r_k + (1-\rho_2)g.g$
8: Modified first and second moments: $\hat{s} = \frac{s_{k+1}}{1-{\rho_1}^k}, \hat{r} = \frac{r_{k+1}}{1-{\rho_2}^k}$
9: Use the first-order moment as the gradient, use the second-order moment to adjust the learning speed, and update the parameters: $\theta_{k+1} = \theta_k - \frac{\alpha}{\sqrt{\hat{r}}+\delta}\hat{s}$
10: k=k+1
11: **Until** $k <= K$ or θ converges.

The method uses the flow processing framework of Python, Tensorflow, to facilitate the adjustment of the algorithm strategy. The training set generates 64-bit floating-point numbers within the specified range according to the domain Ω of the nonlinear system.

5. APPLICATIONS

In this section, to analyze the efficiency of the aforementioned methods the following two ordinary differential equations are used. A comparison of all methods is also given in the form of tables, presented here.

4.6. Example 1

Consider the following first order nonlinear problem associated with initial conditions:

$$\begin{cases} u'(x) + u^2(x) = 1, \\ u(0) = 0, x \geq 0 \end{cases} \tag{10}$$

That is the dimensionless equation of a free-falling sphere in air, and the exact solution of (10) is obtained as $u(x) = \frac{1-e^{-2x}}{1+e^{-2x}}$. The n-th order of approximation solution by HAM to equation (10) is obtained according to equation (4) at $n = 4$, which depends on the convergence control parameter c_0. Table 1, is devoted to the effectiveness of the five approaches discussed for determining convergence control parameter c_0.

Here, "ASR" denotes the average square residual error, and "CPU- time" is elapsed time of the algorithm execution in Mathematica software.

Table 1. Summary of Conclusions on Comparison of Methods

Method	c_0	**ASR**	**CPU-time"(seconds)"**
Yabushita [7]: Exact square residual error	-0.764403587912440	$5.07268183014275 \times 10^{-9}$	2.3191520
Liao [8]: Average square residual error	-0.752906134386632	$6.45375575203388 \times 10^{-9}$	0.0168476
Abbasbandy [9]: L _2-norm method	-0.749434375807557	$7.33213223469420 \times 10^{-9}$	0.0172573
Turkyilmazoglu [10]: Optimum value from the ratio	-0.778082766013357	$8.24473289640082 \times 10^{-9}$	0.0156286
Tonghui [11]: Mini batch gradient descent (MBGD)	-0.753752138818738	$6.26969587358416 \times 10^{-9}$	0.0215954

4.7. Example 2

Consider the initial value problem of third order nonlinear ordinary differential equation:

$$\begin{cases} u'''(x) + u(x)u''(x) - u'^2(x) - 3u'(x) = 0, \\ u(0) = 0, u'(0) = 1, u''(0) = -2, x \geq 0 \end{cases} \tag{11}$$

The exact solution of (11) is obtained as $u(x) = \frac{1}{2}(1 - e^{-2x})$. The n-th order of approximation solution by HAM to equation (11) is obtained according to equation (4) at $n = 6$, which depends on the convergence control parameter c_0. Table 2, is devoted to the effectiveness of the five approaches discussed for determining convergence control parameter c_0. Here, "ASR" denotes the average square residual error, and "CPU- time" is elapsed time of the algorithm execution in Mathematica software.

Table 2. Summary of Conclusions on Comparison of Methods

Author: Method	c_0	**ASR**	**CPU-time"(seconds)"**
Yabushita [7]: Exact square residual error	-0.993203890219743	$9.09494701772928 \times 10^{-12}$	3.0863082
Liao [8]: Average square residual error	-1.076077466507653	$8.07631295174360 \times 10^{-10}$	0.0510978
Abbasbandy [9]: L _2-norm method	-0.993176092855073	$3.63797880709171 \times 10^{-12}$	0.0840526
Turkyilmazoglu [10]: Optimum value from the ratio	-1.076166376442172	$8.07631295174360 \times 10^{-10}$	0.0157218
Tonghui [11]: Mini batch gradient descent (MBGD)	-0.99584779684031	$1.81898940354585 \times 10^{-12}$	0.1168406

6. CONCLUSIONS

The Homotopy Analysis Method (HAM) is a powerful analytical tool for solving nonlinear equations, renowned for its ability to construct accurate approximate solutions. A key factor in the effectiveness of HAM is the determination of the convergence control parameter, which plays a crucial role in ensuring solution stability, accuracy, and efficiency. This study has provided a comprehensive review of five distinct approaches for selecting the optimal convergence control parameter, analyzing their performance, advantages, and limitations. These approaches include traditional methods like constant h-curves and residual error minimization, as well as advanced techniques involving optimization and algorithmic enhancements.

The findings emphasize that proper parameter selection not only improves the accuracy of HAM solutions but also extends its applicability to a broad range of problems in fields such as fluid mechanics, thermal dynamics, epidemic modeling, and quantum systems. The parameter selection algorithms discussed here are significant for advancing HAM's utility, providing a pathway for more efficient and reliable solutions to complex nonlinear problems.

Moreover, these methods offer promising prospects for further development. For instance, integrating artificial intelligence and machine learning to dynamically optimize the convergence control parameter could enhance HAM's adaptability and robustness. This study, therefore, lays a strong foundation for future research aimed at refining HAM and broadening its applications, making it a critical resource for solving nonlinear equations in diverse scientific and engineering contexts.

Declaration

We acknowledge that we used ChatGPT to enhance the academic writing of our manuscript while ensuring the originality and integrity of our work.

Transparency Statement

The data supporting this study are available upon reasonable request to the corresponding author, subject to ethical and confidentiality considerations.

Acknowledgments

We would like to express our gratitude to all individuals who contributed to this project.

Declaration of Interest

The authors declare that they have no competing interests.

Funding

This research received no specific grant from any funding agency, commercial, or not-for-profit sectors.

REFERENCES

[1] Abbasbandy, S., Hashemi, M. S., & Hashim, I. (2013). On convergence of homotopy analysis method and its application to fractional integro-differential equations. Quaestiones Mathematicae, 36(1), 93-105. https://doi.org/10.2989/16073606.2013.780336.

[2] Turkyilmazoglu, M. (2016). An effective approach for evaluation of the optimal convergence control parameter in the homotopy analysis method. Filomat, 30(6), 1633-1650. https://doi.org/10.2298/FIL1606633T.

[3] El-Tawil, M., & Huseen, S. N. (2013). On convergence of q-Homotopy Analysis Method. The International Journal of Contemporary Mathematical Sciences, 8(11), 481-497. https://doi.org/10.12988/ijcms.2013.13048

[4] Hassan, H. N., & Rashidi, M. M. (2014). An analytic solution of micropolar flow in a porous channel with mass injection using homotopy analysis method. International Journal of Numerical Methods for Heat & Fluid Flow, 24(2), 419-437. https://doi.org/10.1108/HFF-08-2011-0158.

[5] Yin, X., Kumar, S., & Kumar, D. (2015). A modified homotopy analysis method for solution of fractional wave equations. Advances in Mechanical Engineering, 7, 1-12. https://doi.org/10.1177/1687814015620330.

[6] Akinyemi, L., Şenol, M., & Huseen, S. N. (2021). Modified homotopy methods for generalized fractional perturbed Zakharov-Kuznetsov equation in dusty plasma. Advances in Difference Equations, 2021(1), 1-27. https://doi.org/10.1186/s13662-020-03208-5.

[7] Yabushita, K., Yamashita, M., & Tsuboi, K. (2007). An analytic solution of projective motion with the quadratic resistance law using homotopy analysis method. Journal of Physics A: Mathematical and Theoretical, 40, 8403-8416. https://doi.org/10.1088/1751-8113/40/29/015.

[8] Liao, S. J. (2010). An optimal homotopy-analysis approach for strongly nonlinear differential equations. Communications in Nonlinear Science and Numerical Simulation, 15, 2003-2016. https://doi.org/10.1016/j.cnsns.2009.09.002.

[9] Abbasbandy, S., & Jalili, M. (2013). Determination of optimal convergence-control parameter value in homotopy analysis method. Numerical Algorithms, 64, 593-605. https://doi.org/10.1007/s11075-012-9680-9

[10] Tonghui, Z., & Yinping, L. (2022). Determination of convergence control parameters in homotopy analysis solutions based on machine learning technique. Journal of East China Normal University (Natural Science), 2022(2), 34.

[11] Tajaddodianfar, F., Yazdi, M., & Pishkenari, H. N. (2017). Nonlinear dynamics of MEMS/NEMS resonators: Analytical solution by the homotopy analysis method. Microsystem Technologies, 23(6), 1913-1926. https://doi.org/10.1007/s00542-016-2947-7.